

Multi-level Monte Carlo Method and its Numerical Analysis in Black-Scholes Option Model

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Background

Standard Monte Carlo Method

- Flexible
- Independent of dimensions
- **Estimation Variance**
- **Convergence Speed**

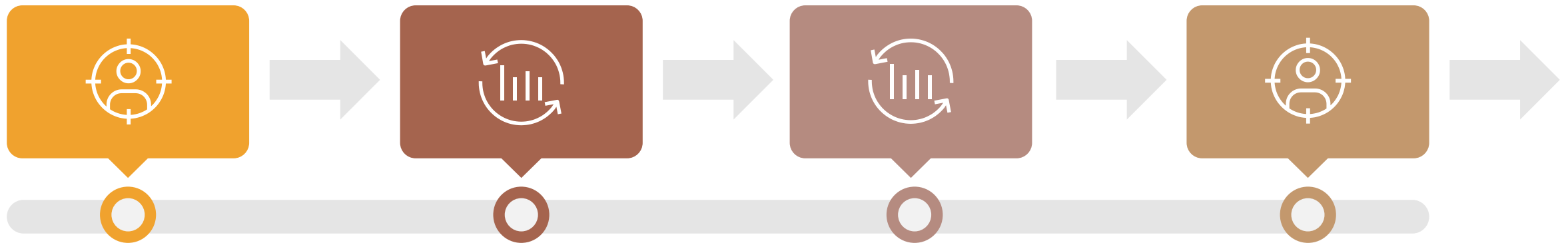
Problem

How many samples needed?

**I want accuracy and don't want
too much cost. How?**

Objectives: Estimated variance, Convergence Speed, Definition of
Cost...

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Objectives

Advantage of MLMC,
compared with MC

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Mathematical Derivation: MC,
MLMC

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Scholes Option Pricing

Results Analysis

Simulation Accuracy

Simulation Costs

Conclusions

MLMC on improving
efficiency

MLMC on reducing cost

Methodology

1. Mathematical Derivation of MLMC

Brownian Motion, Ito Integral, SDE→

→Standard MC, Two-level MC, MLMC

2. Numerical Analysis(EM method)

Standard MC, MLMC

on European, Asian and lookback Options

1. Brownian Motion

A Randomly Moving Suspended Particle:

$$W_{t+\Delta t} = W_t + \alpha * N(0, \Delta t)$$

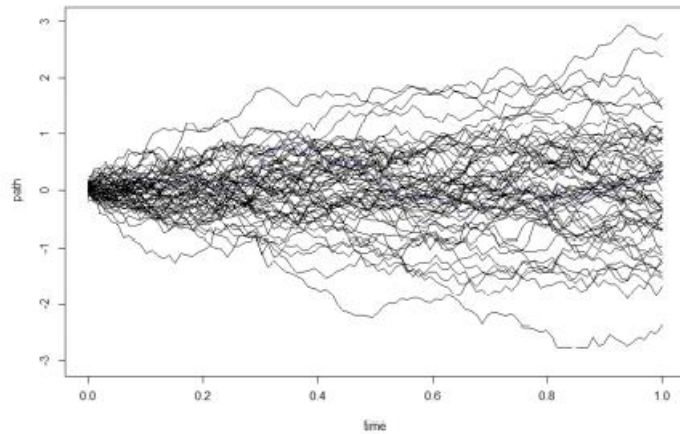


Figure 2.2: 60 paths of standard Brownian Motion(dt=0.01)

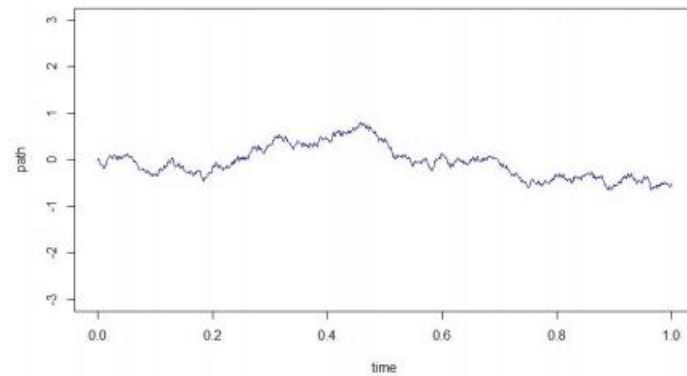


Figure 2.3: Average of 60 paths of standard Brownian Motion

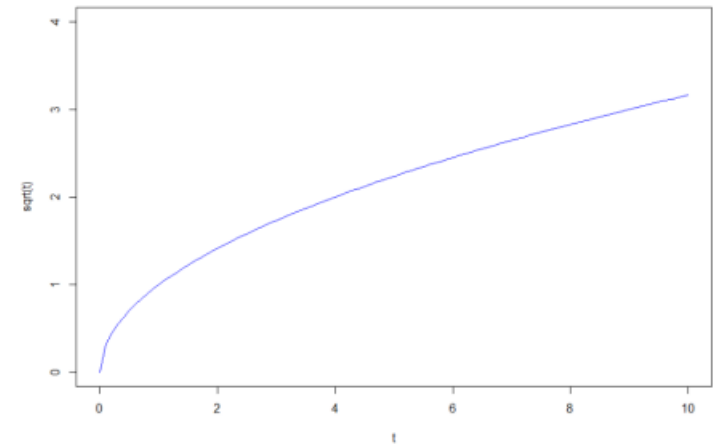


Figure 2.4: Expected Variance of standard Brownian Motion

Definition

Definition 1 [2]

A scalar standard Brownian motion, or standard Wiener process, over $[0, T]$ is a random variable $W(t)$ that depends continuously on $t \in [0, T]$ and satisfies the following three conditions.

- 1. $W(0) = 0$ (with probability 1).*
- 2. For $0 \leq s < t \leq T$ the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$; equivalently, $W(t) - W(s) \sqrt{t - s} \sim N(0, 1)$, where $N(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance.*
- 3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.*

2.SDE and Ito Integral

Difference from ODE

$$\frac{dX_t}{dt} = f(X_t, t)$$

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t) * \text{noise}$$

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t) * \Delta W_t$$

2.SDE and Ito Integral

Ito Integral

$$\frac{dX_t}{dt} = f(X_t, t) + g(X_t, t) * \Delta W_t$$

$$X_t = X_0 + \int_0^t f(X_s, s) ds + \int_0^t g(X_s, s) dW_s$$

$$I = \int_0^t g(X_s, s) * N(0, dt)$$

$$I = \lim_{N \rightarrow \infty} \sum_{n=1}^N g(X_{t_n}, t_n) * N\left(0, \frac{t}{N}\right) \quad t_n = \frac{tn}{N}$$

3. Standard MC

Step1: Find a distribution density function $f_X(x)$, where $\int_a^b f_X(x)dx = 1$, in order to facilitate calculations, Uniform Distribution is generally used;

Step2: Assign the function g according to the probability density to obtain a new function

$$g^*(x) = \frac{g(x)}{f_X(x)} \Rightarrow I = \int_a^b g^*(x) f_X(x) dx;$$

Step3: According to Step1, let X_i obey Uniform Distribution, that is $X_i (i = 1, 2, \dots, N) \Rightarrow f_X(x) = \frac{1}{b-a}$, where X_i are MC sample points;

Step4: Find the new function g^* 's average of the values at each sample point (i.e. $I \approx \hat{I} = \frac{1}{N} \sum_{i=1}^N g^*(X_i)$).

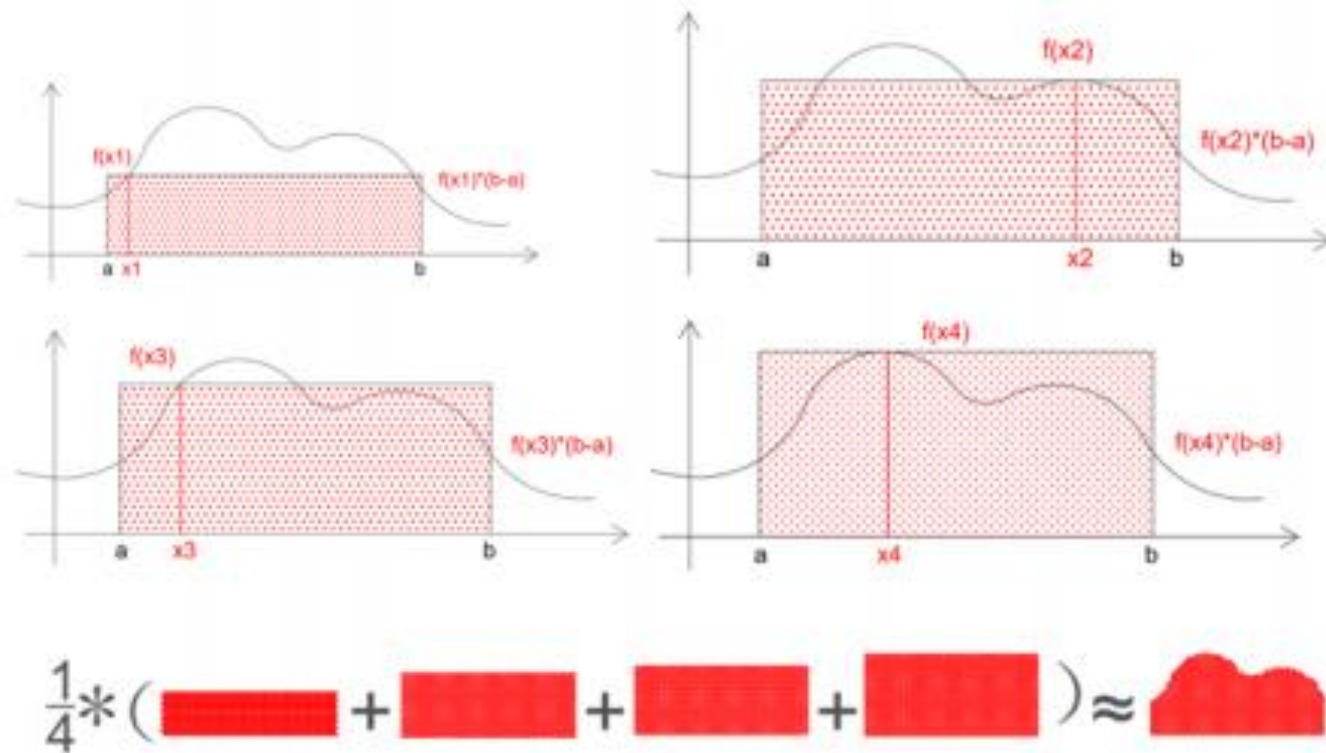


Figure 2.7: Process of MC for Integral

4. Two-level MC and MLMC

Two-level:

$$E[Q_1] = E[Q_0] + E[Q_1 - Q_0]$$

$$E[Q_1] \approx \frac{1}{N_0} \sum_{n=0}^{N_0} Q_0^{(n)} + \frac{1}{N_1} \sum_{n=0}^{N_1} (Q_1^{(n)} - Q_0^{(n)})$$

Simulation Cost

$$N_0 C_0 + N_1 C_1$$

Simulation Variance

$$\frac{N_1}{N_0} = \sqrt{\frac{V_1}{C_1}} \sqrt{\frac{V_0}{C_0}}$$

Multi-level:

$$E[Q_L] \approx \frac{1}{N_0} \sum_{i=1}^{N_0} Q_0(\omega_{i,0}) + \sum_{l=1}^L \left\{ \frac{1}{N_l} \sum_{i=1}^{N_l} (Q_l(\omega_{i,l}) - Q_{l-1}(\omega_{i,l})) \right\}$$

$$\begin{aligned} \text{Or } E[g(X_T)] &= E[g_L(\hat{X}_T)] \\ &= (E[g_0(X_T)] + \sum_{l=1}^L E[(g_l(\hat{X}_T) - g_{l-1}(\hat{X}_T))]) \end{aligned}$$

$$\sum_{l=0}^L (N_l^{-1} V_l + \lambda^2 N_l C_l) \quad (\text{Lagrange Multiplier}) \quad \lambda = \epsilon^{-2} (\sum_{l=0}^L \sqrt{V_l C_l})$$

Simulation Cost

$$C = \sum_{l=0}^L N_l C_l = \sum_{l=0}^L \lambda \sqrt{\frac{V_l}{C_l}} C_l = \epsilon^{-2} \sum_{l=0}^L \sqrt{V_l C_l} \sum_{l=0}^L \sqrt{V_l C_l} = \epsilon^{-2} (\sum_{l=0}^L \sqrt{V_l C_l})^2$$

$$C = O(\text{TOL}^{-2} (\log \text{TOL})^2)$$

Simulation Variance

$$\min_{N_l} \text{Var}(A_{ML}) = \sum_{l=0}^L \frac{V_l}{N_l}$$

5. Experiment: Option Pricing

Estimation Error

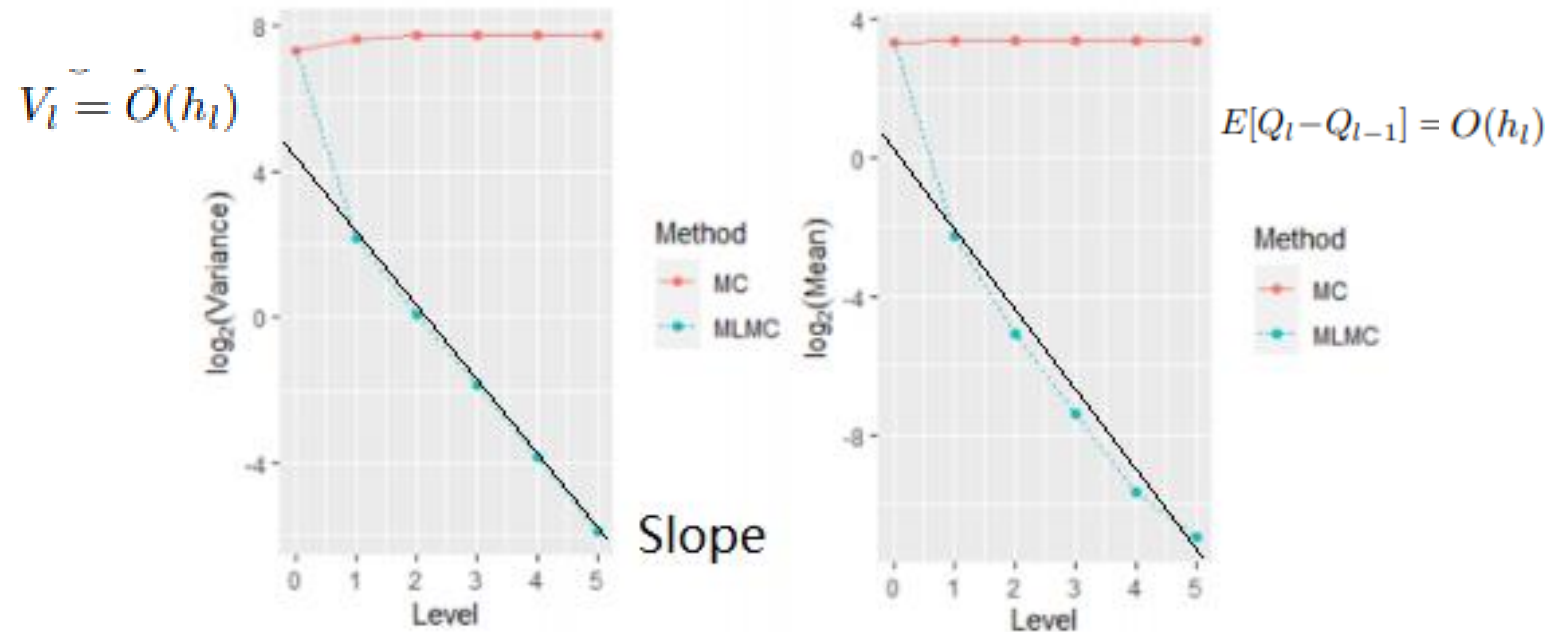


Figure 3.1: Variance/Mean to level 1 on European Options

5. Experiment: Option Pricing

Convergence Speed

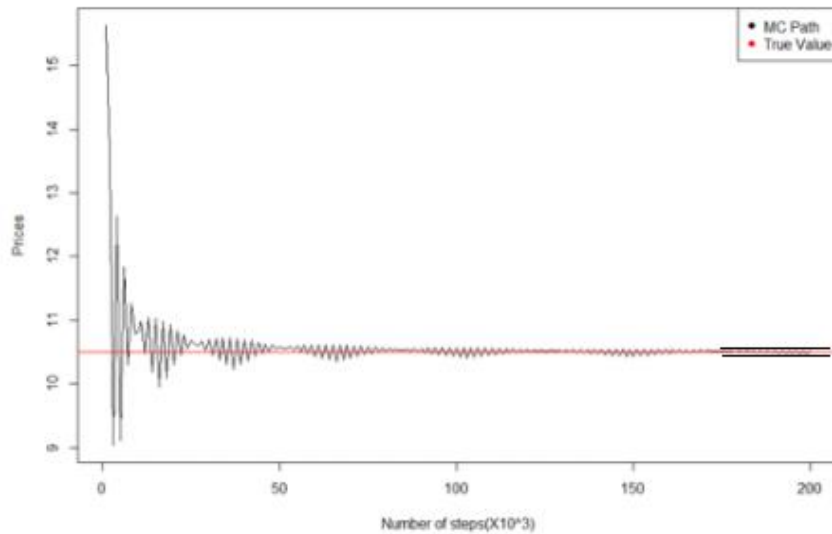
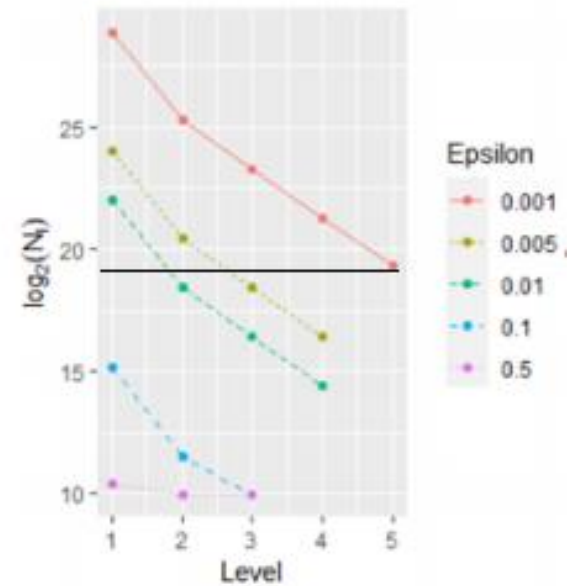


Figure 3.6: MC Convergence to Final Result



6. Conclusions

- Estimation Error
- Convergence Speed

MLMC is Better!

Future Direction

Uncertainty

$$\text{MSE} = O(\epsilon^2)$$

Initial Samples N_0

Thank You